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Measurement of the dynamic elastic constants of short isotropic cylinders

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Abstract

A method based on a single test is proposed to characterize the elasticity of an isotropic homogeneous material in the shape of a cylinder of any slenderness (length–diameter) ratio. Firstly, the Rayleigh–Ritz method is used to determine the natural frequencies of the cylinders vibrating axisymmetrically. The study is focused on cylindrical samples with diameter and length of similar magnitude so that the shear modulus and the Poisson ratio can be calculated simultaneously. Subsequently, the theoretical results for cylinders of slenderness ratio between 0.1 and 3 are analyzed in order to obtain the data required to determine the elastic constants from one of the two lowest measured natural frequencies and their quotient. The analysis of the results demonstrates that any slenderness ratio is useful in the calculation of the elastic constants, although in some cases the third natural frequency should be used. Furthermore, the influence of the length–diameter quotient on the sensitivity of the method is analyzed by evaluating the systematic uncertainties for both dynamic elastic constants. Finally, the method is experimentally tested by characterizing two steel cylinders with slenderness ratios 0.1 and 1, respectively. The results demonstrate that uncertainties for both Poisson ratio and the shear modulus are smaller when the slenderness ratio is 1.

1. Introduction

The value of the two elastic constants of isotropic materials can be determined by static and dynamic measurements. Dynamic methods based on elastic wave measurements have been increasingly used to determine accurately the “adiabatic” elastic constants. Traditional resonance

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methods make use of two experiments on a bar: in one of them a torsional vibration is excited and in the other either longitudinal or flexural vibration is induced.

One method, simpler than others from a theoretical point of view, is based on the study of the vibrations of a system free of external disturbances during the time the system remains under study. This has an advantage since there are no exciting forces and the mathematical equations involved are therefore simpler.

A very simple sample is a cylinder of length L and diameter D. It is simple due to its manufacture, its manipulation and the implied mathematical calculation. Let v and G be its Poisson ratio and shear modulus, respectively.

The elementary theory of vibration of free bars assumes that the length of a bar is much greater than its diameter. Hence, the longitudinal natural frequency f_N, density ρ and Young's modulus E=2(1 + v)G are related by the formula $f_N = N(E/\rho)^{1/2}/(2L)$, where N is an integer. The natural frequency is said to correspond to a symmetric mode if N is odd and to an antisymmetric mode if N is even. Such natural frequencies depend on only one elastic property, Young's modulus. Therefore, the measurement of f_N permits the calculation of E, but not of both elastic constants. In cylinders whose lengths are of the same order of magnitude as their diameters, a contraction or an extension causes non-negligible radial displacements, which depend on v. Therefore, natural frequencies will depend on the two elastic constants of the material, on two geometrical characteristics of the test piece such as L and D, and on the density of the cylinder.

It is necessary to apply the equations of elasticity and the boundary conditions to obtain exact natural frequencies. For a finite cylinder it is impossible to obtain analytical solutions to the equations of motion which satisfy the boundary conditions. Thus, approximate solutions are sought. One of the most widely used procedures for finding such solutions is the Rayleigh–Ritz method. Although this method is applicable to all continuous systems, expressions satisfying all boundary conditions of a problem are difficult to find except for simple geometrical shapes. Fortunately, a cylinder is a simple shape when cylindrical co-ordinates are used. There is much research which applies this technique to the study of vibrations of cylinders. The displacement functions traditionally used are trigonometric or Bessel functions in certain cases [1,2] and power series in others [3,4]. In this paper, power series are selected. As usual, for simplicity of formulation, the non-dimensional frequency $\Omega = \pi f D(\rho/G)^{1/2}$ is used, where f is the ordinary frequency measured in Hz.

The finite element method is based on the same variational principles. Among other papers applying this method to vibration analysis, the paper presented by Larsson [5] is representative; the purpose of his work is to investigate numerically and experimentally the in-plane modes of a free isotropic rectangular plate. Finite element simulations have also been used [6] to verify the accuracy of a direct method of determining the flexural stiffness of vibrating anisotropic plates.

The interest in using variational principles to derive the eigenvalue equation is due to the connection between natural frequencies and elastic constants. The Rayleigh–Ritz method was used in pioneering work [7] to calculate resonance frequencies, and the results obtained were used to determine the elastic constants by means of the well-known cube-resonance method. Other authors [8,9] have refined this methodology and applied the rectangular parallelepiped resonance method to the elastic characterization of crystals. Resonant ultrasound techniques have also been used to determine the elastic constants of cylinders [10] and parallelepiped-shaped composites [11]. In all these cases, approximate techniques are first used to solve the equations, then experimental measurements of the natural frequencies are carried out. Calculation of elastic constants is based on the determination of the differences between theoretical and experimental resonance frequencies; a function of such differences is minimized to yield optimal estimates for elastic constants.

The objective of this paper is to provide a method to determine the elastic constants of isotropic materials. The methodology is based on obtaining the spectrum of free vibration of a cylindrical sample which has been set in vibration by an axially applied impact. The lowest axisymmetric natural frequencies are the input data used to determine both elastic constants. The Rayleigh–Ritz method is used to find a numerical solution for the free vibration of a finite cylinder. However, this work differs from those cases referred to above in both the theoretical and experimental procedures which determine the elastic constants. In the aforementioned methods, the elastic constants are computed by minimizing a function which depends on the difference between the computed and measured natural frequencies. Thus, an iterative process should be followed. On the other hand, the data provided in this paper enable v and G to be determined directly from the two lowest measured natural frequencies for most of the short cylinders, and therefore no further calculations are required.

The experimental resonance frequencies in the papers referred to are usually determined using two piezoelectric transducers: an emitter and a receiver. Forced vibration is induced to find the response to a harmonic excitation. Thus, a sweeping process is followed to obtain the voltage–frequency output signal, whose maxima are the resonances. In this work, after a mechanical impact the sample is left to vibrate on its own and the free vibration is detected using a laser interferometer with a broadband frequency which permits, in a single test, the determination of the vibration spectrum. Thus, there is practically no interaction with the vibrating sample.

From the Ω versus v table provided by Leissa and So [4] for $L/D = 1$, it was observed that the elastic constants can be determined. Therefore, in a previous article [12], a methodology was proposed for the calculation of the two elastic constants of an isotropic cylindrical sample of $L/D = 1$, which consists of a single experiment for measuring the two lowest natural frequencies f_1 and f_2 from the axisymmetric vibration. The knowledge of the values of Ω_1 and Ω_2 for diverse values of v permits the calculation of the elastic constants by subsequent trials. More recently [13] this method has been improved. In this renewed method, it is sufficient to know the quotient f_2/f_1 to obtain v directly by using a precise numerical table, from which G is obtained independently of v from quotient f_2/f_1 and the lowest natural frequency.

This paper improves on the previous paper in as much as it is not limited to the case of $L/D = 1$, since any slenderness ratio is considered. However, mainly small values of L/D are explored. Furthermore, the uncertainties of v and G are determined in order to find the most favourable slenderness ratio.

2. Calculation of non-dimensional frequencies

As indicated, if the quotient L/D is large, the longitudinal natural frequencies of a cylinder depend on elastic constant E but not on v. Therefore, for any large slenderness ratios the method proposed for the calculation of both elastic constants by means of a single test is expected to be inefficient. Therefore vibrations of cylinders whose quotients L/D are small should be studied. The study will be focused mainly on the domain $0.1 \leqslant L/D \leqslant 3$. Although values of the Poisson ratio in the interval $-1 < v < 0.5$ are permissible, most ordinary materials have values in the interval $0 < v < 0.5$; here the field of variability is limited from 0 to 0.49999.

The Rayleigh–Ritz method allows values for the free natural frequencies of a cylinder to be obtained on the basis of permissible displacement functions. When considering only the modes with axial symmetry or axisymmetric modes, the azimuthal variable θ does not appear and, in addition, the displacements have only two non-zero components, radial u and axial w. Therefore the torsional axisymmetric vibrations are excluded. The analysis is focused on the free vibrations of a cylinder, i.e., the bases and the lateral surface are free and there are no bulk forces. By using non-dimensional co-ordinates: r, the radial co-ordinate divided by cylinder radius; and z, the axial co–ordinate divided by cylinder length, the assumed displacements in the cylinder are

$$u(r, \theta z, t) = U(r, z)\sin(\omega t), \tag{1}$$

$$w(r, \theta z, t) = W(r, z)\sin(\omega t),$$

where ω is the natural angular frequency, $\omega = 2\pi f$, and t is time. Let the origin of the co-ordinates and the OZ-axis be the centre of the cylinder and its axis, respectively (see Fig. 1).

Let the displacements be expressed by means of algebraic polynomials. Since all the surfaces of the cylinder are free, suitable test functions are

$$U(r, z) = \sum_{i=1}^{I}\sum_{j=0}^{J} A_{ij} r^i z^i \quad \text{and} \quad W(r, z) = \sum_{p=0}^{P}\sum_{q=0}^{Q} C_{pq} r^p z^q, \tag{2}$$

where $i = 0$ is left out in order to avoid singularities in the stresses at $r = 0$. If displacements are symmetric with respect to the central cross-section, the longitudinal displacements are odd functions of variable z, whereas the displacements in the radial direction are even functions of z.

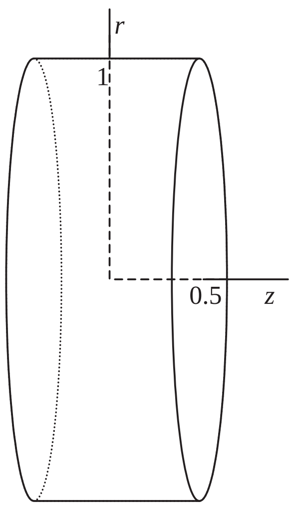

Fig. 1. The sample and the non-dimensional cylindrical co-ordinate system. The ratio r of the distance to the revolution axis and the cylinder radius ranges from 0 to 1. The ratio z of the distance to the straight section plane and the sample length ranges from –0.5 to 0.5.

These modes of vibration correspond to those called symmetric axisymmetric modes. On the other hand, when even functions of variable z are used for w, and odd functions of z for u, the solutions found are designated as antisymmetric axisymmetric modes. In symmetric modes, j takes only even values and q odd values whereas in antisymmetric modes, j takes odd values and q even.

The expressions of the functionals of maximum potential energy and maximum kinetic energy in a period of axisymmetric vibration are

$$V_{\max} = 2\pi GL \int_{-1/2}^{1/2} \int_0^1 \left\{ \frac{\nu}{(1-2\nu)} \left(\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{R}{L}\frac{\partial W}{\partial z} \right)^2 + \left(\frac{\partial U}{\partial r} \right)^2 + \left(\frac{U}{r} \right)^2 + \left(\frac{R}{L}\frac{\partial W}{\partial z} \right)^2 + \frac{1}{2}\left(\frac{R}{L}\frac{\partial U}{\partial z} + \frac{\partial W}{\partial r} \right)^2 \right\} r\,\mathrm{d}r\,\mathrm{d}z, \tag{3}$$

$$T_{\max} = \pi\rho L R^2 \omega^2 \int_{-1/2}^{1/2} \int_0^1 \left(U^2 + W^2 \right) r\,\mathrm{d}r\,\mathrm{d}z.$$

The minimizing conditions imply that $\partial(V_{max}-T_{max})/\partial A_{ij}=0$, $\partial(V_{max}-T_{max})/\partial C_{pq}=0$, for all values of i, j, p and q. These conditions constitute a set of linear homogenous algebraic equations in the unknown quantities A_{ij} and C_{pq}. The requirement of non-trivial solutions gives the admissible values for Ω^2. For each eigenvalue Ω^2, the set of linear equations supplies the eigenvectors, whose components are the unknown quantities A_{ij} and C_{pq}. Since the best possible convergence is desirable, as many polynomial terms are included in the calculation as the computer can deal with. For $L/D=0.1$, the term type $r^{10}\ z^5$ has been achieved and, for $L/D=3$, term $r^5\ z^9$. Hence convergence of the results for the lowest non-dimensional frequencies is expected; in effect, all the digits of the non-dimensional frequencies shown in the tables remain unchanged when raising or lowering the degree of the polynomial by one unit.

The results obtained by means of the numerical calculation of the non–dimensional frequency are complicated since Ω depends on both ν and L/D. The numerical calculation yielded [14] the five lowest symmetric non-dimensional frequencies Ω_{s1}, Ω_{s2}, …, Ω_{s5}, and the five antisymmetric ones Ω_{a1}, Ω_{a2}, …, Ω_{a5}, for each value of L/D=0.1, 0.2, …, 3.0 and for at least 11 values of ν. Certain results corresponding to $L/D=0.1$ are listed in Table 1, whereas Table 2 refers to $L/D=1$. The result for $L/D=1$ may be compared with those obtained by Leissa and So [4, Tables 3 and 4], although the comparison is only partial since only four significant digits are given in this reference, whereas six are given here; however, the present results rounded to four digits agree with those presented by them. Henceforth, let Ω_1 be the lowest value of Ω independent of symmetry type, Ω_2 the second lowest value, and so on.

The purpose of this work is to use the results of numerical calculations to determine elastic constants, therefore we have exhaustively analyzed such results in order to provide the data required to characterize the elasticity of the material. This study appears in the following section.

3. Numerical results. Methodology to determine the elastic constants

Due to the limitation of the bandwidth of the excitation system, the spectrum of the vibration presents greater amplitude for the lowest natural frequencies, which are expected to be detected

Table 1
Non-dimensional frequencies $\Omega = \pi f D\sqrt{\rho/G}$ for the two lowest frequency axisymmetric modes of a free cylinder with $L/D=0.1$, their ratio and the magnitudes B, H, K and M for the calculation of the systematic uncertainties

ν	Ω_1	Ω_2	Ω_2/Ω_1	B	H	K	M
0.00	0.64232	2.58745	4.02829	15.2	9.3	30.1	33.5
0.02	0.65309	2.61340	4.00159	15.4	9.5	30.1	33.8
0.04	0.66405	2.63999	3.97559	15.6	9.8	30.0	34.2
0.06	0.67521	2.66724	3.95024	15.7	10.2	30.0	34.4
0.08	0.68660	2.69517	3.92539	16.0	10.6	30.1	34.8
0.10	0.69821	2.72383	3.90116	16.1	10.8	30.0	35.2
0.12	0.71007	2.75325	3.87743	16.2	11.2	30.0	35.5
0.14	0.72220	2.78345	3.85413	16.3	11.4	30.0	35.8
0.16	0.73462	2.81449	3.83122	16.4	11.8	29.9	36.1
0.18	0.74734	2.84640	3.80871	16.5	12.1	29.9	36.4
0.20	0.76039	2.87923	3.78652	16.5	12.4	29.9	36.7
0.22	0.77379	2.91302	3.76461	16.6	12.7	29.9	37.1
0.24	0.78756	2.94783	3.74299	16.6	13.0	29.9	37.3
0.26	0.80173	2.98371	3.72159	16.5	13.2	29.8	37.5
0.28	0.81634	3.02071	3.70031	16.5	13.4	29.8	37.8
0.30	0.83140	3.05891	3.67923	16.4	13.5	29.8	38.0
0.32	0.84697	3.09837	3.65818	16.2	13.8	29.8	38.2
0.34	0.86307	3.13916	3.63720	16.1	13.8	29.7	38.4
0.36	0.87974	3.18137	3.61626	15.9	13.8	29.7	38.5
0.38	0.89704	3.22507	3.59524	15.7	13.9	29.6	38.5
0.40	0.91501	3.27036	3.57412	15.4	13.9	29.6	38.6
0.42	0.93371	3.31735	3.55287	15.1	13.8	29.6	38.6
0.44	0.95320	3.36615	3.53142	14.7	13.6	29.5	38.5
0.46	0.97356	3.41687	3.50967	14.4	13.5	29.5	38.4
0.48	0.99485	3.46966	3.48762	14.7	13.2	29.4	38.2
0.49999	1.01603	3.52185	3.46629	—	—	—	—

primarily. Thus, the analysis of the results will be centred on the identification and later use of these lowest natural frequencies to characterize the elasticity of the material of the cylinder.

For a specific cylinder of a given material, the spectrum of its natural frequencies is obtained experimentally, and the natural frequencies are denoted in ascending order by f_1, f_2 and f_3. Proportionality of Ω and f is the result of the definition of non–dimensional frequency, i.e., the order of the natural frequencies and that of Ω are equivalent. Analogously, the quotients are proportional to the ratio of the natural frequencies, i.e., $\Omega_j/\Omega_i=f_j/f_i$ for all values of i and j.

In order to apply the method proposed to determine ν and G from the two lowest natural frequencies, it is essential, for the tested cylinder, that ν be a single-valued function of the ratio Ω_2/Ω_1. Higher detected natural frequencies can be used in problematic cases or to verify results.

The calculation of the quotient Ω_2/Ω_1 for each value of ν, presented below, has been made for slenderness ratio values in the interval $0.1\leqslant L/D\leqslant 3$, with increments in L/D of 0.1. The quotient Ω_2/Ω_1 versus the Poisson ratio ν appears in Fig. 2 as curves for the indicated quotients of L/D. Fig. 2(a) presents these curves for values of the slenderness ratio between 0.1 and 1. High values of the quotient Ω_2/Ω_1 are observed for the ratio $L/D=0.1$, which corresponds to a thin circular disc.

Table 2
Non-dimensional frequencies $\Omega = \pi f D\sqrt{\rho/G}$ for the two lowest frequency axisymmetric modes of a free cylinder with $L/D = 1$, their ratio and the magnitudes B, H, K and M for the calculation of the systematic uncertainties

ν	Ω_1	Ω_2	Ω_2/Ω_1	B	H	K	M
0.00	2.22144	2.30623	1.03817	5.7	3.0	12.2	3.1
0.02	2.24160	2.34391	1.04564	4.7	2.5	12.2	1.3
0.04	2.25795	2.38173	1.05482	4.1	2.2	12.1	0.3
0.06	2.27109	2.41967	1.06542	3.7	2.0	12.1	0.4
0.08	2.28162	2.45772	1.07718	3.5	1.9	12.1	0.8
0.10	2.29012	2.49583	1.08982	3.4	1.8	12.0	1.1
0.12	2.29704	2.53399	1.10315	3.4	1.7	12.0	1.4
0.14	2.30274	2.57215	1.11700	3.3	1.7	11.9	1.5
0.16	2.30749	2.61027	1.13122	3.3	1.6	11.9	1.7
0.18	2.31149	2.64831	1.14572	3.3	1.6	11.8	1.8
0.20	2.31489	2.68621	1.16041	3.4	1.6	11.8	1.9
0.22	2.31782	2.72393	1.17521	3.4	1.6	11.8	2.0
0.24	2.32037	2.76141	1.19007	3.5	1.6	11.8	2.0
0.26	2.32259	2.79858	1.20494	3.6	1.6	11.7	2.1
0.28	2.32456	2.83540	1.21976	3.7	1.6	11.7	2.2
0.30	2.32630	2.87178	1.23448	3.8	1.6	11.7	2.2
0.32	2.32786	2.90768	1.24908	3.9	1.6	11.7	2.3
0.34	2.32926	2.94303	1.26350	4.0	1.6	11.7	2.3
0.36	2.33053	2.97777	1.27772	4.2	1.6	11.6	2.4
0.38	2.33168	3.01185	1.29171	4.3	1.7	11.6	2.4
0.40	2.33273	3.04521	1.30543	4.5	1.7	11.6	2.4
0.42	2.33369	3.07780	1.31886	4.7	1.7	11.6	2.5
0.44	2.33457	3.10960	1.33198	4.9	1.8	11.6	2.5
0.46	2.33539	3.14055	1.34476	5.1	1.8	11.6	2.5
0.48	2.33614	3.17064	1.35721	5.3	1.8	11.6	2.6
0.49999	2.33684	3.19984	1.36930	—	—	—	—

Fig. 2(b) presents such curves for values of the slenderness ratio between 1 and 3. From Fig. 2, it is deduced that ν is a single-valued function of the quotient Ω_2/Ω_1 for slenderness ratios $L/D = 0.10$, 0.14, 0.16, 0.18, 0.2, 1, 1.1, 1.2, 1.3, and for $L/D \geqslant 1.9$.

For $L/D = 0.1$, numerical calculations show that $\Omega_{a1} < \Omega_{a2} < \Omega_{s1}$ for all values of the Poisson ratio. In Fig. 2(a), it is observed that ν is a single-valued and decreasing function of Ω_2/Ω_1 with a pronounced slope. Therefore, quotient f_2/f_1 allows the direct calculation of the Poisson ratio. The values of Ω_1, Ω_2 and Ω_2/Ω_1 are given in detail in Table 1. The Ω_2/Ω_1 quotient ranges from 4.02829 to 3.46629, where the interval length is 0.56200. Thus, for $L/D = 0.1$, the interval of existence of Ω_2/Ω_1 is the largest among the cases studied.

For $L/D = 0.2$, the following arrangement for the values of Ω: $\Omega_{a1} < \Omega_{s1} < \Omega_{a2} < \Omega_{a3} < \Omega_{s2} < \Omega_{a4}$ is found. The Poisson ratio is a monotonic increasing function of the quotient $\Omega_2/\Omega_1 = \Omega_{s1}/\Omega_{a1}$. The quotient $f_3/f_1 = \Omega_{a2}/\Omega_{a1}$ also directly gives the value of ν, however this is not shown in this article. Furthermore, the latter quotient presents a more pronounced slope as opposed to the former, hence a higher sensitivity for the calculation of ν is expected in this case.

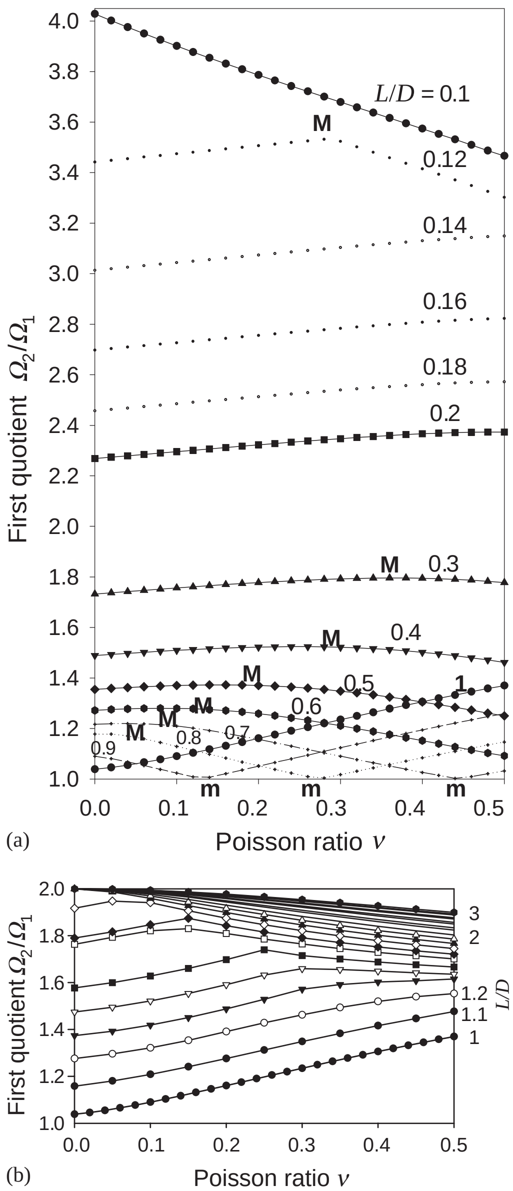

Fig. 2. The quotient Ω_2/Ω_1, which is f_2/f_1, versus the Poisson ratio v for cylinders with slenderness ratio L/D: (a) from 0.1 to 1. Note the maxima (M) of the curves for L/D from 0.3 to 0.8 (and 0.12), and the minima (m), which are equal to the unit, corresponding to the cases in which the first two modes intersect (from 0.7 to 0.9); (b) from 1 to 3. There are maxima for L/D between 1.4 and 1.8.

To study the change in sign of the slope from $L/D = 0.1$ to 0.2, intermediate values of the slenderness ratio were analysed. The results are shown in Fig. 2(a).

The cases of L/D from 1 to 1.3, Fig. 2(b), are very similar. For $L/D = 1$, 1.1, 1.2, Ω_2/Ω_1 is equal to Ω_{a1}/Ω_{s1}, and for $L/D = 1.3$, Ω_2/Ω_1 is equal to Ω_{s2}/Ω_{s1}.

In Fig. 2(b) similar behaviour is observed for all values $L/D \geqslant 1.9$. It is found that ν is a monotone decreasing function of quotient $\Omega_2/\Omega_1 = \Omega_{a1}/\Omega_{s1}$.

Consequently, the method used in the calculation of the Poisson ratio for cylinders based on the determination of $\Omega_2/\Omega_1 = f_2/f_1$, is directly applicable when this quotient is a single-valued function of ν. In these cases, for a determined ratio L/D, the value of ν can be graphically obtained from the value of Ω_2/Ω_1 (Fig. 2). It will be enough to find the intersection point of the Ω_2/Ω_1 horizontal line and the L/D curve, whose X co-ordinate is the value of ν for the material. If higher precision is desired, tabulated values should be used (included here only for slenderness ratio 0.1 and 1).

Since ν is determined from Ω_2/Ω_1, G is calculated from the same quotient. Tables 1 and 2 give the values of Ω_1 and Ω_2/Ω_1 for $L/D = 0.1$ and 1, respectively. These tables, and those corresponding to other slenderness ratios [14], allows one to obtain Ω_1 from quotient Ω_2/Ω_1. In fact, if properties such as ρ, D and f_1 are measured in the sample, the shear modulus G may be calculated from the equation $\Omega_1 = \pi D f_1 \sqrt{\rho/G}$. Fig. 3 provides graphically estimated values of Ω_1. Fig. 3(a) represents Ω_2/Ω_1 versus Ω_1 for values of slenderness ratio from 0.1 to 1.0, while Fig. 3(b) shows such dependency for values of L/D between 1 and 3. It can be observed that Ω_1, and therefore G, is a single-valued function of Ω_2/Ω_1 for $L/D = 0.1$, 0.14, 0.16, 0.18, 0.2, 1.0, 1.1, 1.2, 1.3 and also for values equal to or greater than 1.9.

In Fig. 3, the curves of $L/D = $ constant are of similar character within these certain ranges of slenderness ratio L/D. Hence, the method proposed to characterize the elasticity of materials is directly applicable to certain sets of neighbouring values of L/D, where only the first two natural frequencies of the spectrum are used.

For $L/D \gg 1$ the elementary theory for slender bars starts to be applicable. For this theory, $\Omega_1 = \pi(1/2 + \nu/2)^{1/2}/(L/D)$ and $\Omega_2/\Omega_1 = 2$. Therefore curves for higher slenderness ratios tend towards:

(a) a family of hyperbolae, in diagram Ω_1 versus L/D,
(b) the horizontal straight line $\Omega_2/\Omega_1 = 2$, in Figs. 2(b) and 3(b).

Hence, for large L/D, the proposed method does not provide accurate values of the elastic constants. Nevertheless the calculation of the Young modulus becomes easier.

For $L/D \ll 1$, rods become plates. The length of the rod may be considered infinitely small with respect to its diameter. Therefore for very short cylinders, only very high natural frequencies are expected for symmetric modes. If plates are considered as two-dimensional elastic systems, antisymmetric modes are expected, where bending moments and transverse shear forces are active. The governing equation in terms of polar coordinates is given by [15]

$$\frac{Eh^2}{12(1-\nu^2)}\nabla^4 w + \rho\frac{\partial^2 w}{\partial t^2} = \frac{q}{h}, \tag{4}$$

where h is the plate depth, and q the external force per unit area. If there are no surface forces, $q = 0$, Love [16] argues that the natural frequencies for the normal modes are proportional to the thickness, and inversely proportional to the square of the linear dimension. For the present disc $f_a \sim L/D^2$, therefore, $\Omega_a \sim fD \sim L/D$ and Ω_a versus L/D is a straight line in accordance with our numerical calculation [14]. For short cylinders, $L/D \to 0$, therefore Ω_a and f_a are small. Hence the lowest natural frequencies appearing in an experimental spectrum correspond to antisymmetric

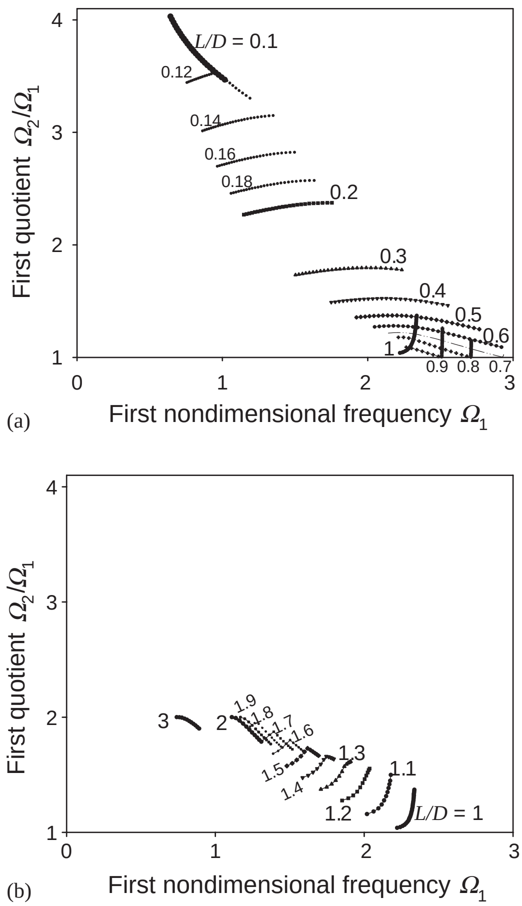

Fig. 3. The quotient Ω_2/Ω_1 as a function of the non-dimensional frequency Ω_1, for values of L/D: (a) included in the interval from 0.1 to 1; (b) from 1 to 3. For a given cylinder, Ω_1 determines the shear modulus G, in accordance with $G=\pi^2 f^2 D^2 \rho \Omega_1^{-2}$.

modes. Furthermore, only the elastic property $E/(1-\nu^2)$ appears in Eq. (4); therefore, only this elastic property may be determined by wave experiments with plates. Hence very short cylinders are not adequate for the calculation of the elastic constants in a single experiment. Since the present methodology can be applied to cylinders with $L/D=0.1$, only cylinders whose slenderness ratio $L/D \ll 0.1$ have such a limitation.

4. Singular slenderness ratios

Except for the slenderness ratios indicated in the previous section, ν is not a single-valued function of Ω_2/Ω_1, and therefore higher natural frequencies should be used. In the interval studied the singular slenderness ratios are 0.12, from 0.3 to 0.9 and from 1.4 to 1.8. For these values of the slenderness ratio, a detailed study has been performed, which is described in full by Nieves [14].

The results provide the lowest natural frequencies to be used in determining v. A summary of such a study is shown in Fig. 4, where the required information to calculate v from one of the quotients Ω_2/Ω_1, Ω_3/Ω_1, Ω_3/Ω_2 is given.

From the aforementioned study it is concluded that the method proposed may also be applied to characterize cylinders with singular slenderness ratios, where the third natural frequency will be used. In each situation it is necessary to use the applicable quotient. The data of Fig. 4, together with those of Ω_1 from either Fig. 3 or computed tables lead to the determination of the elastic constants of the samples.

5. Sensitivity based on the slenderness ratio

The Poisson ratio is a function of Ω_2/Ω_1 and L/D, $v = v(\Omega_2/\Omega_1, L/D)$. In order to calculate v with high sensitivity, the partial derivative of v with respect to the quotient Ω_2/Ω_1 should be as small as possible. However such a derivative depends on v and L/D. Thus, the purpose here is to quantify the best length/diameter ratio which leads to the highest accuracy of calculation.

Now apply the systematic uncertainty methodology. If a physical magnitude y is a function $y = F(\{x_i\})$ of a set of physical magnitudes x_i which have been measured directly and are affected

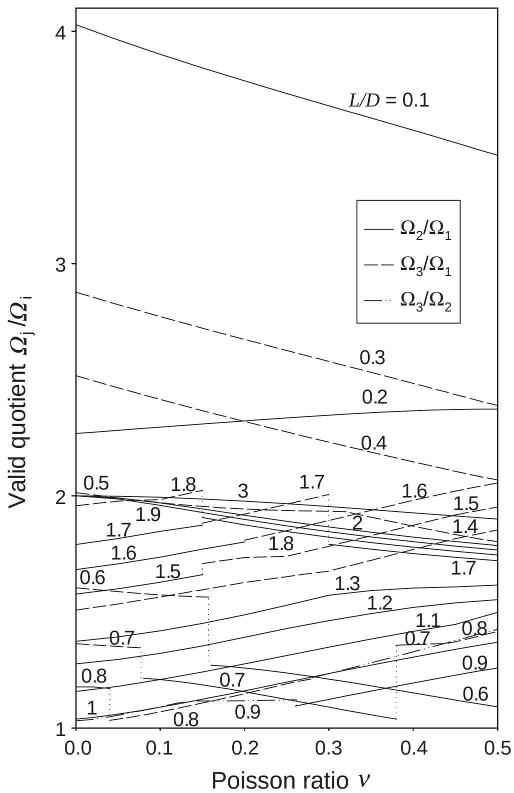

Fig. 4. Single–valued functions of the quotients Ω_j/Ω_i versus v for the adequate lowest natural frequencies and for various values of L/D.

by their respective uncertainties U_{x_i}, then the uncertainty of an indirect measurement U_y is estimated by means of the differential of this function using the absolute values of the partial derivatives [17], that is, $U_y = \sum |\partial F/\partial x_i| \; U_{x_i}$. It will be supposed that all the measuring instruments are well calibrated, therefore their uncertainties U_{x_i} are only due to their sensitivities. First, this method will be applied to the Poisson ratio and then to the shear modulus.

Quotient Ω_2/Ω_1 equals the ratio f_2/f_1 of the frequencies f_2 and f_1 measured directly with a Fourier analyser. Therefore, the uncertainty of v is

$$U_v = \left|\frac{\partial v}{\partial(\Omega_2/\Omega_1)}\right|\left(\frac{U_{f_2}}{f_1} + \frac{f_2 U_{f_1}}{f_1^2}\right) + \left|\frac{\partial v}{\partial(L/D)}\right|\left(\frac{U_L}{D} + \frac{L U_D}{D^2}\right). \tag{5}$$

The values f_1 and f_2 are simultaneously obtained as a result of the FFT analysis when a single experiment is carried out. Thus, the sampling frequency is, according to the Nyquist criterion, at least double the highest natural frequency, i.e., f_2. Therefore, if the sampling theorem is strictly applied, it will be $U_{f1} = U_{f2} = 2f_2/N$, where N is the number of samples which the analogue-to-digital converter acquires. If the apparatus used to measure L and D is the same, $U_D = U_L$. Hence

$$U_v \approx \frac{2}{N}\frac{\Delta v}{\Delta(\Omega_2/\Omega_1)}\left(\frac{\Omega_2^2}{\Omega_1^2} + \frac{\Omega_2}{\Omega_1}\right) + \frac{\Delta v}{\Delta(L/D)}\left(1 + \frac{L}{D}\right)\frac{U_L}{D}, \tag{6}$$

where Δv is the difference between two consecutive input data of v used for the calculation of Ω.

Denote

$$B \equiv \frac{\Delta v}{\Delta(\Omega_2/\Omega_1)}\left(\frac{\Omega_2^2}{\Omega_1^2} + \frac{\Omega_2}{\Omega_1}\right) \quad \text{and} \quad H \equiv \frac{\Delta v}{\Delta(L/D)}\left(1 + \frac{L}{D}\right). \tag{7}$$

The first factor of B may be obtained from the slope of the lines of constant L/D of Fig. 2, and its second factor, the magnitude in brackets, is given by the corresponding ordinate axis. In the same way, the magnitude $\Delta v/\Delta(L/D)$ and, consequently, the value of H can be calculated from the intersection points of the horizontal line for Ω_2/Ω_1 with the L/D curves. Looking at the figures, lines of constant L/D with high slope and low ratio Ω_2/Ω_1, i.e., those lines placed at the bottom, have small values of B. Small values of H are given when the slenderness ratios are small and the lines of L/D = constant are very close. The minimum value of $2B/N + HU_L/D$ will give the best accuracy for v. This condition seems to be fulfilled for cylinders whose slenderness ratio and quotient of the two lowest frequencies are small. However, both the slope $\Delta(\Omega_2/\Omega_1)/\Delta v$ and the quotient $\Delta(L/D)/\Delta v$ should yield high values. Fig. 2(a) shows that the lowest ratio studied, $L/D = 0.1$, has the highest slope, although for this ratio the quotient of the two lowest frequencies is high. Furthermore, $L/D = 1$ has the lowest quotient Ω_2/Ω_1 from among the single-valued functions studied and lines of constant L/D are closer than for L/D=0.1 The value of L/D for which U_v is the minimum appears to be close to one of these two slenderness ratios. Therefore, comparison of U_v for such ratios is carried out.

Table 1 gives the values of B and H for L/D=0.1. If one knew the value of v of the material with which the cylinder is made, the first term of the systematic uncertainty of v would be obtained immediately by reading the B value in its row and multiplying it by $2/N$. However, v is not known; indeed, it has to be sought. Hence, care should be taken and the most unfavourable circumstance should be considered. The maximum value of B which appears in the Table, 16.6, corresponds to the most unfavourable case. This value multiplied by $2/N$ is the maximum of the first term of the systematic uncertainty of v. The same table shows the magnitude H. The partial derivative appearing in H has been calculated for each Ω_2/Ω from both $L/D=0.1$ and 0.099. This H multiplied by U_L/D leads to a precise estimation of the second term in U_v. The maximum value of H is 13.9.

Table 2 gives the values of B and H for L/D=1. Note that the maximum values for B and H which appears in this Table are 5.7 and 3.0, respectively.

In relation to the systematic uncertainty of v, the slenderness ratio L/D=1 is more suitable than L/D=0.1, since its values for B and H are the smallest for all v.

As far as the shear modulus is concerned, from the definition of Ω, it is deduced that

$$G=\frac{\pi^2 f^2 D^2 \rho}{\Omega^2}=\frac{4\pi f^2 m}{L\Omega^2}, \tag{8}$$

where m is the mass of the cylinder. The magnitudes m, L and f are measured directly in the laboratory and Ω_1 is obtained from the tables which relate Ω_1 to Ω_2/Ω_1 and to L/D. Starting from

$$\mathrm{d}\Omega_1=\frac{\partial\Omega_1}{\partial(\Omega_2/\Omega_1)}\frac{\Omega_2}{\Omega_1}\frac{\mathrm{d}f_2}{f_2}-\frac{\partial\Omega_1}{\partial(\Omega_2/\Omega_1)}\frac{\Omega_2}{\Omega_1}\frac{\mathrm{d}f_1}{f_1}+\frac{\partial\Omega_1}{\partial(L/D)}\frac{L}{D}\frac{\mathrm{d}L}{L}-\frac{\partial\Omega_1}{\partial(L/D)}\frac{L}{D}\frac{\mathrm{d}D}{D}, \tag{9}$$

the systematic uncertainty of G is

$$U_G\approx G\frac{U_m}{m}+G\left[\frac{1}{L/D}+\frac{2}{\Omega_1}\left(1+\frac{L}{D}\right)\frac{\Delta\Omega_1}{\Delta(L/D)}\right]\frac{U_L}{D}+\frac{4G}{N}\left[\frac{\Omega_2}{\Omega_1}+\frac{1}{\Omega_1}\left(\frac{\Omega_2^2}{\Omega_1^2}+\frac{\Omega_2}{\Omega_1}\right)\frac{\Delta\Omega_1}{\Delta\left(\Omega_2/\Omega_1\right)}\right], \tag{10}$$

where $\Delta\Omega_1$ is the increment of Ω_1, i.e., the difference between the values in two consecutive rows in the tables. Denote

$$K\equiv\frac{1}{L/D}+\frac{2}{\Omega_1}\left(1+\frac{L}{D}\right)\frac{\Delta\Omega_1}{\Delta(L/D)} \quad \text{and} \quad M\equiv 2\frac{\Omega_2}{\Omega_1}+\frac{2}{\Omega_1}\left(\frac{\Omega_2^2}{\Omega_1^2}+\frac{\Omega_2}{\Omega_1}\right)\frac{\Delta\Omega_1}{\Delta\left(\Omega_2/\Omega_1\right)}. \tag{11}$$

For the value of K to be small, Ω_1 must be high and the curves of Fig. 3 must be close to each other. The value of M is smaller for low values of Ω_2/Ω_1 and steep slopes of $\Delta(\Omega_2/\Omega_1)/\Delta\Omega_1$. The ratio $L/D=1$ has the highest Ω_1 among the ratios having single-valued functions together with the lowest Ω_2/Ω_1. Let us also compare U_G for the ratios studied. Table 1 gives K and M for $L/D=0.1$ and Table 2 gives the K and M values for $L/D=1$.

In order to simplify the comparison between the systematic uncertainties of the shear modulus of two cylinders of different slenderness ratios, the following reasonable hypotheses will be considered:

(1) The relative uncertainty of a balance depends upon its quality and not on the measurable mass. Therefore U_m/m is practically independent of the cylinder size.
(2) A similar conclusion can be drawn for U_L/D.

Therefore, the relative systematic uncertainty of the shear modulus depends, apart from the quality of the balance, the quality of the ruler, and the memory depth of the data acquirer, on the slenderness ratio and the quotient f_2/f_1. The dependence on the two latter quantities is expressed by means of the magnitudes K and M. However, when comparing the appropriateness of two cylinders in order to calculate G with minimum relative uncertainty, it is enough to compare the sum of the second and third terms in (10), $KU_L/D + 2M/N$. The cylinder with the minimum of this sum is the most suitable cylinder for the measurement of G.

Thus, from Table 1, which refers to $L/D = 0.1$, it is deduced that the maximum value of K is 30.1 and the maximum value of M is 38.6. From Table 2 for $L/D = 1$, the respective values $K = 12.2$ and $M = 3.1$ are deduced for these uncertainties. These results indicate that the most suitable value for the slenderness ratio is the $L/D = 1$, at least for these 2 cylinders. A $L/D = 1$ cylinder is also recommended for the calculation of the Poisson ratio.

For $L/D = 1.0$, 1.1, 1.2, and 1.3, Fig. 2(b) and Fig. 3(b) show decreasing slopes and increasing values of the quotient of the frequencies. Hence $L/D = 1.0$ seems to be the best slenderness ratio to calculate the elastic constants by means of the quotient of the first two natural frequencies.

Fig. 2 shows some maxima in the lines of constant L/D. In these maxima$\partial v/\partial(\Omega_2/\Omega_1) = \infty$, hence $U_v = \infty$. Therefore the calculation of v by f_2/f_1 is not possible. It seems reasonable that, close to the maxima, the calculation of the Poisson ratio by f_2/f_1 gives inaccurate values.

Analogue considerations on the Ω_2/Ω_1-Ω_1 plane indicate that the points close to the maxima are not suitable to calculate the shear modulus from the two lowest natural axisymmetric frequencies.

For increasing values of L/D above 3, $\Delta\Omega_1/\Delta(\Omega_2/\Omega_1)$ tends to infinity, therefore from Eq. (10) the systematic uncertainty is unbounded and the shear modulus, as expected, is not computable.

6. Experimental tests

The procedure for generating and detecting the vibration of the sample is described in a previous paper [12]. A cylinder is placed horizontally, supported in the centre over a small rubber block, so that it can vibrate almost freely. A small steel sphere measuring 3 mm in diameter is used to excite vibration of the sample by applying a brief axial impact to the centre of the cylinder base. The duration of the impact is estimated to be 10^{-5} s. This type of excitation allows the sample to oscillate freely in its natural modes, since following the impact no additional appreciable forces act upon the sample.

An OP-35 I/O laser interferometer from Ultra Optec Inc., is used to measure the vibration of the sample. With this system, out-of-plane and in-plane displacement components can be detected at the same point, although detection is not simultaneous. Detection is point-like and without contact with the sample. At the same time, the system has a broad bandwidth, from 1 kHz to 35 MHz, allowing simultaneous detection of various vibration modes, with a resolution of approximately 1 nm. A demodulating unit yields a signal proportional to the instantaneous displacement at the detection point. In the present case, the interferometer is operating in the out-of-plane mode. The out-of-plane component is detected at the centre of the rod base opposite to the base where the impact is applied. A TEKTRONIK TDS-430A oscilloscope digitises the resulting signal. The sampling frequency used is $f_s = 250$ kSa/s and the number of samples 10 000. Finally, the FFT of the signal is obtained and the maximum amplitudes in the spectrum correspond to the natural frequencies.

Two DIN 1.4031 stainless steel samples are used to determine the elastic constants and to carry out the experimental tests of different uncertainties based on the slenderness ratio:

(1) The first test cylinder has diameter $D = 49.20$ mm, length $L = 4.92$ mm, therefore $L/D \approx 0.1$; and given mass $m = 73.6$ g, hence its density is $\rho = 7869$ kg/m^3.

Fig. 5 shows the vibration spectrum of the cylinder obtained following the aforementioned procedure. The maxima corresponding to the lowest natural frequencies are at $f_1 = f_{a1} = 16\,700$ Hz and $f_2 = f_{a2} = 61\,375$ Hz, which are used in the determination of ν and G, as described. First, the quotient $\Omega_2/\Omega_1 = f_2/f_1 = 3.67515$ is calculated which, according to Table 1, and by interpolating linearly, corresponds to $\nu = 0.3039$. Hence the value $\Omega_1 = 0.83442$ is obtained from the column Ω_1 by interpolating again. From this value, and those known for f_1, D and ρ, $G = 75.3$ GPa is found.

Furthermore, the systematic uncertainties of ν and G are obtained following the previously described methodology. The systematic uncertainties of the direct measures for the length and mass are $U_L = 10^{-5}$ m, $U_m = 10^{-4}$ kg, respectively. The uncertainty obtained for the Poisson ratio is $U_\nu = 2B/N + HU_L/D = 32.8 \times 10^{-4} + 27.0 \times 10^{-4} = 0.0060$, and its relative uncertainty 2.0%. For the shear modulus, there is a relative uncertainty $U_G/G = 1.36 \times 10^{-3} + 5.96 \times 10^{-3} + 7.60 \times 10^{-3} = 1.5\%$.

(2) The aforementioned methodology and the same experimental devices are used to characterize a $L = D = 49.92$ mm cylinder of the same material. In this sample, the values obtained for the elastic constants are $\nu = 0.2963$ and $G = 76.35$ GPa and the relative uncertainties are estimated as 0.36% for ν and 0.29% for G. The experimental results, $G = 77.6$ GPa and $\nu = 0.283$, calculated from the measurement of v_p and v_s velocities together with the value $G = 76.7$ GPa from the first torsional natural frequency are reported in Ref. [13]. Such results are in close agreement with those given in this paper.

As a result of the calculation of the sensitivity based on the slenderness ratio, it is concluded that the uncertainties are small for $L/D = 1$. One factor yielding an improvement of the uncertainty for case $L/D = 1$ with respect to case $L/D = 0.1$ is the fact that the quotient between the two lowest natural frequencies is three times smaller for the first case than for the second. This explains why B and M are also smaller since both are increasing functions of the quotient Ω_2/Ω_1. Furthermore, the total quotient $\Delta\Omega_1/\Delta(\Omega_2/\Omega_1)$ for $L/D = 0.1$ which is approximately double that for $L/D = 1$ (compare the corresponding curves in Fig. 3(a)) leads to greater uncertainty.

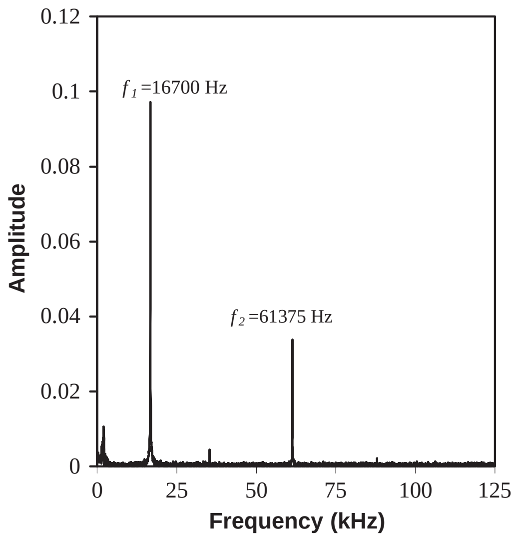

Fig. 5. Frequency spectrum for a steel disc with $L/D = 0.1$.

Moreover, there are other factors which favour the use of the method with $L = D$ cylinders. For instance, working with these samples is easier, since they are supported better than narrow discs with $L/D = 0.1$ and there is less influence from the unavoidable imperfections in the required circular cylindrical form. However, it should not be concluded that the unitary slenderness ratio is the optimal one, due to the lack of studies on the sensitivity of the method for higher natural frequencies.

Experimental instruments have been used for which $2/N = 2 \times 10^{-4}$ and $U_L/D = 2 \times 10^{-4}$ approximately. These values are commonly found in physics laboratories. Therefore, the uncertainties may be expressed as $U_v = 2 \times 10^{-4}(B + H)$ and $U_G/G = U_m/m + 2 \times 10^{-4}(K + M)$. The magnitudes B, H, K and M may be calculated for each value of interest of L/D, as shown in Tables 1 and 2. Hence $B + H$ and $K + M$ enables one to obtain the optimum L/D, i.e., the L/D with the lowest systematic uncertainties for the Poisson ratio and shear modulus, respectively.

7. Conclusions

A single-test method for the determination of the elastic constants of short isotropic cylinders whose slenderness ratios range from 0.1 to 3 is presented. The proposed methodology, based on the measurement of two axisymmetric natural frequencies, allows one to calculate the Poisson ratio and the shear modulus as follows:

(a) For cylinders with slenderness ratios 0.1, from 0.14 to 0.2, from 1 to 1.3, and from 1.9 to 3, only the two lowest natural frequencies are required. The figures and tables included give the values of v and G, from the quotient of such frequencies together with one of these frequencies.
(b) For slenderness ratios from 0.3 to 0.9 and from 1.4 to 1.8, the third frequency should be obtained since the aforementioned quotient is not a single-valued function. The quotient to be used in these cases, either Ω_3/Ω_1 or Ω_3/Ω_2, is determined and shown graphically (Fig. 4).

The proposed method is shown to be precise by investigating the uncertainties. The expressions given allow the calculation of the uncertainties of v and G for any cylinder. The values of the uncertainties for cylinders whose slenderness ratio is of the order of one seem to be the smallest. Detailed tables showing the terms to be included in the calculations of the uncertainties are given for $L/D = 1$ and 0.1, where it is also shown that the uncertainties depend on the Poisson ratio, i.e., on the material.

Two steel cylinders with slenderness ratio 0.1 and 1 are experimentally tested in order to compare the results. The difference between the values calculated by the method shown and other types of measures is negligible and is explained by the uncertainties. Uncertainties smaller than 0.4% are found for $L/D = 1$ for both elastic constants.

Thus, it seems that $L/D = 1$ is the best value for the application of this method.

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